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Classical integrable two-dimensional models inspired by SUSY quantum mechanics

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Abstract. A class of integrable two-dimensional (2D) classical systems with integrals of motion of fourth order in momenta is obtained from the quantum analogues with the help of deformed SUSY algebra. With similar technique a new class of potentials connected with the Lax method is found which provides the integrability of corresponding 2D Hamiltonian systems. In addition, some integrable 2D systems with potentials expressed in elliptic functions are explored.

1. Introduction

Construction of classical integrable systems with additional integrals of motion is of considerable interest in mathematical physics (see [1] and references therein). Multidimensional integrable systems play an important role in describing the dynamics analogously to one-dimensional (1D) manifestly integrable systems. In particular, they may serve as zero approximations of perturbation theory in the case of weak, nonintegrable perturbations. A variety of traditional approaches to this problem exists starting from Kepler, Kowalewski until the Lax method. On the other hand, a modern viewpoint on how to build classical integrable systems is based on the symmetries of related quantum systems [2]. Recently, a method for searching quantum-integrable two-dimensional (2D) systems was developed [3, 4] with the help of a deformed supersymmetry (SUSY) algebra formed by intertwining differential operators of finite order. Supersymmetry [5–7], i.e. the construction of the isospectral pair of Hamiltonians, was proved [3] to be in one-to-one correspondence to integrability of both Hamiltonians, i.e. to existence of a differential symmetry operator, which is polynomial in derivatives and which transforms solutions of the 2D Schrödinger equation into other solutions with the same energy. Quasiclassical reduction of the deformed SUSY algebra [3] gave the factorization of classical integrals of motion for the corresponding Hamiltonians [8]. As a result, the structure of analytically resolved integrals of motion became clearer, and new classes of integrable potentials were found [8, 9].

In our paper we continue our study [9] of classical systems in which integrability is induced or inspired by a deformed SUSY algebra for the relevant quantum systems. The concise basic construction of systems possessing a dynamical symmetry with the help of higher-derivative SUSY algebra is essentially supplemented with algorithms for searching analytical solutions of related nonlinear equations for coefficients of functions of symmetry operators and potentials.

In section 2 the intertwining relations between a pair of quantum Schrödinger-type Hamiltonians for general differential operators $q^{\pm}$ of second order are investigated. The class of particular solutions of these relations is constructed for the cases of hyperbolic (Lorentz) $g_{ik} = \mathrm{diag}(1, -1)$ and degenerate $g_{ik} = (1, 0)$ metric structures of operators $q^{\pm}$ in second derivatives. The differential operators of fourth order in derivatives, which are symmetry operators for intertwined Hamiltonians, are built. In section 3 the classical limit $\hbar \to 0$ for the Hamiltonians is considered, and the class of systems with integrals of motion of fourth order in momenta is obtained. In section 4 a new class of integrable systems with potentials connected to the Lax method is derived using ansätze and techniques taken from 2D SUSY quantum mechanics (section 2). Section 5 is devoted to a description of some integrable systems expressed in elliptic functions. We stress that quite a few of the obtained potentials do not allow separation of variables in known coordinate systems and some of them so far have not been found.

2. Quantum integrable 2D systems

In the two-dimensional generalization [3, 4, 8] of higher-order SUSY quantum mechanics [10] the intertwining relations of second order in derivatives are most essential:

$$\begin{aligned} &H^{(1)}q^{+} = q^{+}H^{(2)} \qquad q^{-}H^{(1)} = H^{(2)}q^{-} \\ &H^{(i)} = -\hbar^2\Delta + V(\vec{x}) \qquad \Delta \equiv \partial_1^2 + \partial_2^2 \qquad \partial_i \equiv \partial/\partial x_i \\ &q^{+} = (q^{-})^{\dagger} = \hbar^2 g_{ik}(\vec{x})\partial_i\partial_k + \hbar C_i(\vec{x},\hbar)\partial_i + B(\vec{x},\hbar) \end{aligned} \tag{1}$$

where $\hbar$ is Planck's constant and all coefficient functions are real.

This means that, up to zero modes of $q^{\pm}$, spectra of $H^{(i)}$ coincide and their eigenfunctions are

$$\Psi^{(2)} \sim q^{-}\Psi^{(1)} \qquad \Psi^{(1)} \sim q^{+}\Psi^{(2)}. \tag{2}$$

The intertwining relations (1) lead to existence of the symmetry operators $R^{(1)}$, $R^{(2)}$ for the Hamiltonians $H^{(1)}$, $H^{(2)}$, respectively,

$$[R^{(i)}, H^{(i)}] = 0 \qquad R^{(1)} = q^{+}q^{-} \qquad R^{(2)} = q^{-}q^{+} \qquad i = 1, 2. \tag{3}$$

In the 1D case [10] analogous symmetry operators $R^{(i)}$ become polynomials of $H^{(i)}$ with constant coefficients. The distinguishing peculiarity of the 2D case is existence [4] of nontrivial dynamical symmetry operators $R^{(i)}$ which are not reduced to functions of the Hamiltonians $H^{(i)}$.

It was shown in [3] that for the unit metrics $g_{ik} = \delta_{ik}$ operators $R^{(i)}$ can be written as second-order differential operators (up to a function of $H^{(i)}$) and corresponding quantum systems allow separation of variables in parabolic, elliptic or polar coordinate systems. For all other metrics g_{ik} operators $R^{(i)}$ are of fourth order in the derivatives.

The intertwining relations (1) are equivalent to the following system of differential equations:

$$\begin{aligned} &\hbar\partial_i C_k + \hbar\partial_k C_i + \hbar^2\Delta g_{ik} - (V^{(1)} - V^{(2)})g_{ik} = 0 \\ &\hbar^2\Delta C_i + 2\hbar\partial_i B + 2\hbar g_{ik}\partial_k V^{(2)} - (V^{(1)} - V^{(2)})C_i = 0 \\ &\hbar^2\Delta B + \hbar^2 g_{ik}\partial_k\partial_i V^{(2)} + \hbar C_i\partial_i V^{(2)} - (V^{(1)} - V^{(2)})B = 0 \end{aligned} \tag{4}$$

where the metrics g_{ik} is a quadratic polynomial in x_1, x_2:

$$\begin{aligned} &g_{11} = ax_2^2 + a_1x_2 + b_1 \\ &g_{22} = ax_1^2 + a_2x_1 + b_2 \\ &g_{12} = -\tfrac{1}{2}(2ax_1x_2 + a_1x_1 + a_2x_2) + b_3. \end{aligned}$$

2.1. Lorentz metrics

For the supercharges with Lorentz metrics ($g_{ik} = \mathrm{diag}(1, -1)$):

$$q^+ = \hbar^2(\partial_1^2 - \partial_2^2) + \hbar C_k \partial_k + B \tag{5}$$

a solution of (4) can be reduced [4] to a solution of the system

$$\partial_-(C_- F) = -\partial_+(C_+ F) \tag{6}$$

$$\partial_+^2 F = \partial_-^2 F \tag{7}$$

where $C_1 \mp C_2 \equiv C_\pm(x_\pm)$ depend only on $x_\pm$, respectively. Equation (7) means that the function F can be represented as a sum $F = F_1(x_+ + x_-) + F_2(x_+ - x_-)$. The potentials $V^{(1,2)}$ and the function B are expressed in terms of solutions of the system (6) and (7):

$$V^{(1,2)} = \pm\tfrac{1}{2}\hbar(C'_+ + C'_-) + \tfrac{1}{8}(C_+^2 + C_-^2) + \tfrac{1}{4}(F_2(x_+ - x_-) - F_1(x_+ + x_-)) \tag{8}$$

$$B = \tfrac{1}{4}(C_+C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)). \tag{9}$$

The solutions for functions F, which admit additionally the factorization $F = F_+(x_+) \cdot F_-(x_-)$, were found in [14]. In the present paper other solutions of (6) and (7) will be built.

(1) After substitution of the general solution of (6)

$$F = L\left(\int \frac{\mathrm{d}x_+}{C_+} - \int \frac{\mathrm{d}x_-}{C_-}\right)\Big/(C_+C_-) \tag{10}$$

into (7), we obtain the functional-differential equation for functions L and $A'_\pm \equiv 1/C_\pm(x_\pm)$:

$$\left(\frac{A'''_+}{A'_+} - \frac{A'''_-}{A'_-}\right)L(A_+ - A_-) + 3(A''_+ + A''_-)L'(A_+ - A_-) + (A'^2_+ - A'^2_-)L''(A_+ - A_-) = 0 \tag{11}$$

where L' denotes the derivative of L with respect to its argument. Equation (11) can be easily solved for functions $A_\pm$ such that $A''_\pm = \lambda^2 A_\pm$, then

$$L(A_+ - A_-) = \alpha(A_+ - A_-)^{-2} + \beta$$

for $A_\pm = \sigma_\pm \exp(\lambda x_\pm) + \delta_\pm \exp(-\lambda x_\pm)$ with $\sigma_+ \cdot \delta_+ = \sigma_- \cdot \delta_-$ and α, β real constants. For $\lambda^2 > 0$ we obtain (up to an arbitrary shift in $x_\pm$) two solutions:

(1a) $A_\pm = k \sinh(\lambda x_\pm)$

(1b) $A_\pm = k \cosh(\lambda x_\pm)$.

Then (10) leads to:

(1a) $$F_1(2x) = F_2(2x) = \frac{k_1}{\cosh^2(\lambda x)} + k_2 \cosh(2\lambda x) \qquad C_\pm = \frac{k}{\cosh(\lambda x_\pm)} \qquad k \neq 0 \tag{12}$$

(1b) $$F_1(2x) = -F_2(2x) = \frac{k_1}{\sinh^2(\lambda x)} + k_2 \sinh^2(\lambda x) \qquad C_\pm = \frac{k}{\sinh(\lambda x_\pm)} \qquad k \neq 0. \tag{13}$$

For $\lambda^2 < 0$ hyperbolic functions must be substituted by trigonometric ones.

At last, in the limiting case of $\lambda = 0$ the solutions have the form

$$F_1(2x) = -F_2(2x) = k_1 x^{-2} + k_2 x^2 \qquad C_\pm = \frac{k}{x_\pm} \qquad k \neq 0 \tag{14}$$

$$F_1(2x) = -F_2(2x) = k_1 x^2 + k_2 x^4 \qquad C_\pm = \pm\frac{k}{x_\pm} \qquad k \neq 0. \tag{15}$$

(2) To find another class of solutions of the system (6) and (7) it is useful to replace in (10) $C_\pm$ by $f_\pm$, such that $C_\pm \equiv \pm f_\pm/f'_\pm$. Then F in (10) is represented in the form $F = U(f_+f_-)f'_+f'_-$ with an arbitrary function U. After substitution in (7) one obtains the equation

$$(f_+'^2 f_-^2 - f_+^2 f_-'^2)U''(f) + 3f\left(\frac{f''_+}{f_+} - \frac{f''_-}{f_-}\right)U'(f) + \left(\frac{f'''_+}{f'_+} - \frac{f'''_-}{f'_-}\right)U(f) = 0 \qquad f \equiv f_+f_-.$$

One can check that $f_\pm = \alpha_\pm \exp(\lambda x_\pm) + \beta_\pm \exp(-\lambda x_\pm)$ and $U = a + 4bf_+f_-$ are its particular solutions (a, b are real constants). Then functions

$$\begin{aligned}
F_1(x) &= k_1(\alpha_+\alpha_- \exp(\lambda x) + \beta_+\beta_- \exp(-\lambda x)) + k_2(\alpha_+^2\alpha_-^2 \exp(2\lambda x) + \beta_+^2\beta_-^2 \exp(-2\lambda x))\\
-F_2(x) &= k_1(\alpha_+\beta_- \exp(\lambda x) + \beta_+\alpha_- \exp(-\lambda x)) + k_2(\alpha_+^2\beta_-^2 \exp(2\lambda x)\\
&\quad + \beta_+^2\alpha_-^2 \exp(-2\lambda x))\\
C_\pm &= \pm\frac{\alpha_\pm \exp(\lambda x_\pm) + \beta_\pm \exp(-\lambda x_\pm)}{\lambda(\alpha_\pm \exp(\lambda x_\pm) - \beta_\pm \exp(-\lambda x_\pm))}
\end{aligned} \tag{16}$$

are real solutions of the system (6) and (7) if $\alpha_\pm, \beta_\pm$ are real for the case $\lambda^2 > 0$ and $\alpha_\pm = \beta_\pm^*$ for the case $\lambda^2 < 0$.

(3) To find a third class of solutions it is useful to rewrite (6) in terms of the variables $x_{1,2}$:

$$2(F_1(x_1) + F_2(x_2))\partial_1(C_+ + C_-) + F_1'(x_1)(C_+ + C_-) + F_2'(x_2)(C_+ - C_-) = 0.$$

Its solutions are:

(3a)
$$\begin{aligned}
&C_+(x_+) = \sigma_1\sigma_2 \exp(\lambda x_+) + \delta_1\delta_2 \exp(-\lambda x_+) + c\\
&C_-(x_-) = \sigma_1\delta_2 \exp(\lambda x_-) + \sigma_2\delta_1 \exp(-\lambda x_-) + c\\
&F_1(x_1) = 0 \qquad F_2(x_2) = \frac{1}{(\sigma_2 \exp(\lambda x_2) - \delta_2 \exp(-\lambda x_2))^2}
\end{aligned} \tag{17}$$

(3b)
$$C_+(x) = C_-(x) = ax^2 + c \qquad F_1(x_1) = 0 \qquad F_2(x_2) = \frac{4b^2}{x_2^2} \tag{18}$$

(3c)
$$\begin{aligned}
&C_+(x_+) = \sigma_1\sigma_2 \exp(\lambda x_+) + \delta_1\delta_2 \exp(-\lambda x_+)\\
&C_-(x_-) = \sigma_1\delta_2 \exp(\lambda x_-) + \sigma_2\delta_1 \exp(-\lambda x_-)\\
&F_{1,2}(x_{1,2}) = \frac{\nu_{1,2}}{(\sigma_{1,2} \exp(\lambda x_{1,2}) \pm \delta_{1,2} \exp(-\lambda x_{1,2}))^2} \pm \gamma.
\end{aligned} \tag{19}$$

Let us remark that two additional solutions, analogous to (3a) and (3b), can be obtained by replacing $F_1(x_1)$ with $F_2(x_2)$ and vice versa.

After inserting these solutions (12)–(19) into the general formulae for potentials (8), we obtain, correspondingly, the following expressions for potentials (20)–(27):

$$\begin{aligned}
V^{(1,2)} = &\mp\frac{\hbar k\lambda}{2}\left[\frac{\sinh(\lambda x_+)}{\cosh^2(\lambda x_+)} + \frac{\sinh(\lambda x_-)}{\cosh^2(\lambda x_-)}\right] + \frac{k^2}{8}\left[\frac{1}{\cosh^2(\lambda x_+)} + \frac{1}{\cosh^2(\lambda x_-)}\right]\\
&+\frac{1}{4}\left[\frac{k_1}{\cosh^2(\lambda x_2)} - \frac{k_1}{\cosh^2(\lambda x_1)} + k_2\cosh(2\lambda x_2) - k_2\cosh(2\lambda x_1)\right]
\end{aligned} \tag{20}$$

$$\begin{aligned}
V^{(1,2)} = &\mp\frac{\hbar k\lambda}{2}\left[\frac{\cosh(\lambda x_+)}{\sinh^2(\lambda x_+)} + \frac{\cosh(\lambda x_-)}{\sinh^2(\lambda x_-)}\right] + \frac{k^2}{8}\left[\frac{1}{\sinh^2(\lambda x_+)} + \frac{1}{\sinh^2(\lambda x_-)}\right]\\
&-\frac{1}{4}\left[\frac{k_1}{\sinh^2(\lambda x_2)} + \frac{k_1}{\sinh^2(\lambda x_1)} + k_2\cosh(2\lambda x_1) + k_2\cosh(2\lambda x_2)\right]
\end{aligned} \tag{21}$$

$$V^{(1,2)} = \mp\frac{\hbar k}{2}\left(\frac{1}{x_+^2} + \frac{1}{x_-^2}\right) + \frac{k^2}{8}\left(\frac{1}{x_+^2} + \frac{1}{x_-^2}\right) - \frac{1}{4}\left[\frac{k_1}{x_1^2} + \frac{k_1}{x_2^2} + k_2(x_1^2 + x_2^2)\right]. \tag{22}$$

Let us note that the potential (21) with $k_2 = 0$ and the potential (22) were investigated in the literature (cf, for example, [11]).

$$V^{(1,2)} = \mp\frac{\overline{h}k}{2}\left(\frac{1}{x_+^2} - \frac{1}{x_-^2}\right) + \frac{k^2}{8}\left(\frac{1}{x_+^2} + \frac{1}{x_-^2}\right) - \frac{1}{4}\left[k_1(x_1^2 + x_2^2) + k_2(x_1^4 + x_2^4)\right] \tag{23}$$

$$\begin{aligned} V^{(1,2)} = {} & \frac{2\alpha_+\beta_+(1 \mp 8\overline{h}\lambda^2) + \alpha_+^2 \exp(2\lambda x_+) + \beta_+^2 \exp(-2\lambda x_+)}{8\lambda^2(\alpha_+ \exp(\lambda x_+) - \beta_+ \exp(-\lambda x_+))^2} \\ & + \frac{2\alpha_-\beta_-(1 \pm 8\overline{h}\lambda^2) + \alpha_-^2 \exp(2\lambda x_-) + \beta_-^2 \exp(-2\lambda x_-)}{8\lambda^2(\alpha_- \exp(\lambda x_-) - \beta_- \exp(-\lambda x_-))^2} \\ & - \tfrac{1}{4}\big[k_1(\alpha_+\beta_- \exp(2\lambda x_2) + \alpha_-\beta_+ \exp(-2\lambda x_2)) + k_2(\alpha_+^2\beta_-^2 \exp(4\lambda x_2) \\ & + \alpha_-^2\beta_+^2 \exp(-4\lambda x_2)) + k_1(\alpha_+\alpha_- \exp(2\lambda x_1) + \beta_+\beta_- \exp(-2\lambda x_1)) \\ & + k_2(\alpha_+^2\alpha_-^2 \exp(4\lambda x_1) + \beta_+^2\beta_-^2 \exp(-4\lambda x_1))\big] \end{aligned} \tag{24}$$

$$\begin{aligned} V^{(1,2)} = {} & \pm\tfrac{1}{2}\overline{h}\lambda(\sigma_1 \exp(\lambda x_1) - \delta_1 \exp(-\lambda x_1))(\sigma_2 \exp(\lambda x_2) + \delta_2 \exp(-\lambda x_2)) \\ & + \tfrac{1}{8}\big[(\sigma_1^2 \exp(\lambda x_1) + \delta_1^2 \exp(-\lambda x_1))(\sigma_2^2 \exp(\lambda x_2) + \delta_2^2 \exp(-\lambda x_2)) \\ & + 2c(\sigma_1 \exp(\lambda x_1) - \delta_1 \exp(-\lambda x_1))(\sigma_2 \exp(\lambda x_2) - \delta_2 \exp(-\lambda x_2))\big] \\ & + \frac{1}{4(\sigma_2 \exp(2\lambda x_2) - \delta_2 \exp(-2\lambda x_2))^2} \end{aligned} \tag{25}$$

$$V^{(1,2)} = \pm 2\overline{h}ax_1 + \tfrac{1}{4}\left[a^2(x_1^4 + x_2^4 + 6x_1^2x_2^2) + ac(x_1^2 + x_2^2)\right] + \frac{b^2}{x_2^2} \tag{26}$$

$$\begin{aligned} V^{(1,2)} = {} & \pm\tfrac{1}{2}\overline{h}\lambda(\sigma_1 \exp(\lambda x_1) - \delta_1 \exp(-\lambda x_1))(\sigma_2 \exp(\lambda x_2) + \delta_2 \exp(-\lambda x_2)) \\ & + \tfrac{1}{8}(\sigma_1^2 \exp(\lambda x_1) + \delta_1^2 \exp(-\lambda x_1))(\sigma_2^2 \exp(\lambda x_2) + \delta_2^2 \exp(-\lambda x_2)) \\ & + \frac{\nu_2}{(\sigma_2 \exp(2\lambda x_2) - \delta_2 \exp(-2\lambda x_2))^2} - \frac{\nu_1}{(\sigma_1 \exp(2\lambda x_1) + \delta_1 \exp(-2\lambda x_1))^2}. \end{aligned} \tag{27}$$

The Hamiltonians with potentials (20)–(27) possess the symmetry operators $R^{(1)} = q^+q^-$, $R^{(2)} = q^-q^+$, where $q^\pm$ can be obtained by inserting solutions (12)–(19) into (5).

It is necessary to note that there are some singular points of potentials (21)–(27) on the plane (x_1, x_2). Therefore, both the asymptotics of corresponding wavefunctions and their behaviour under the action of supertransformation operators $q^\pm$ (5) and of symmetry operators $R^{(1)}$, $R^{(2)}$ have to be investigated. In particular, for the potentials (23)–(27) the operators $q^\pm$ preserve asymptotics of wavefunctions in the singular points, so the operators $R^{(1)}$, $R^{(2)}$ are physical symmetry operators for these systems. For the potentials (21) and (22) symmetry properties have been discussed previously (see, for example, [11]).

2.2. Degenerate metrics

For the supercharges with degenerate metrics $g_{ik} = \mathrm{diag}(1, 0)$:

$$q^+ = \overline{h}^2\partial_1^2 + \overline{h}C_k\partial_k + B \tag{28}$$

equations (4) lead to

$$\begin{aligned} & C_1(\vec{x}) = -x_2F_1'(x_1) + G_1(x_1); \qquad C_2(\vec{x}) = F_1(x_1) \\ & V^{(1)} = \overline{h}(2G_1' - x_2F_1'') + \tfrac{1}{4}x_2^2(F_1^2)'' - x_2(F_1G_1)' + K_1(x_1) + K_2(x_2) \\ & V^{(2)} = \overline{h}x_2F_1'' + \tfrac{1}{4}x_2^2(F_1^2)'' - x_2(F_1G_1)' + K_1(x_1) + K_2(x_2) \\ & B = -\tfrac{1}{2}\overline{h}(G_1' + x_2F_1'') + \tfrac{1}{2}G_1^2 - \tfrac{1}{2}x_2^2F_1F_1'' + x_2F_1G_1' - K_1(x_1) \end{aligned} \tag{29}$$

where the real functions $F_1(x_1)$, $G_1(x_1)$ and $K_1(x_1)$ are solutions of the following system:

$$-\tfrac{1}{2}\hbar^2 G_1''' + \tfrac{1}{2}\hbar((G_1^2)'' + 2G_1'^2) + G_1K_1' + 2G_1'K_1 - F_1(F_1G_1)' - G_1'G_1^2 = m_1F_1 \qquad (30)$$

$$\tfrac{1}{2}\hbar^2 F_1^{(IV)} - \hbar(F_1'G_1'' + 2G_1'F_1'') - G_1(2G_1'F_1' + G_1''F_1)$$
$$-F_1'K_1' + \tfrac{1}{2}F_1(F_1^2)'' - 2G_1'^2F_1 - 2F_1''K_1 = m_2F_1 \qquad (31)$$

$$\tfrac{1}{4}G_1(F_1^2)''' + F_1'(F_1G_1)'' + 3G_1'F_1F_1'' = m_3F_1 \qquad (32)$$

$$\tfrac{1}{4}F_1'(F_1^2)''' + F_1F_1''^2 = m_4F_1 \qquad (33)$$

and $K_2(x_2)$ is the polynomial of x_2 with constant coefficients:

$$K_2(x_2) = m_0 - m_1x_2 - \tfrac{1}{2}m_2x_2^2 - \tfrac{1}{3}m_3x_2^3 + \tfrac{1}{4}m_4x_2^4.$$

Several particular solutions of equation (33) can be found. The constant function $F_1 = k_1$ is the solution of (33) for $m_4 = 0$. To find other solutions we define the new function $U(F_1)$:

$$U(F_1) = F_1'(x_1) \qquad (34)$$

to decrease the order of the differential equation (33):

$$U'' + \frac{3}{U}U'^2 + \frac{3}{F_1}U' - \frac{2m_4}{U^3} = 0. \qquad (35)$$

Inserting its known solution [17] into (34), the following equation for $F_1(x_1)$ is obtained:

$$\int F_1^{1/2}(m_4F_1^4 + nF_1^2 + k)^{-1/4}\,\mathrm{d}F_1 = x_1 \qquad n, k = \text{constant}. \qquad (36)$$

The integral (36) can be written as a finite combination of elementary functions only in the case when two of the three constants m_4, n, k are zero. Thus the solutions of equation (33) in elementary functions are: $F_1 = k_1$; $F_1 = x_1/n$; $F_1 = \left(\frac{3}{2}\right)^{2/3}k^{1/6}x_1^{2/3}$; $F_1 = m_4^{1/2}x_1^2/4$. Below, for simplicity, we shall consider the solutions with particular values of constants m_4, n, k, while solutions with arbitrary values of these constants will differ by some of the coefficients only. To solve equations (30)–(32) it is useful to consider separately two cases: $G_1 \equiv 0$ and $G_1 \not\equiv 0$. In both cases solutions with $F_1 = x_1$ lead to potentials $V^{(1,2)}$ with separation of variables. Below such solutions will be ignored.

(1) $G_1 = 0$.

In this case the potentials $V^{(1,2)}$ for $F_1 = k_1$ correspond again to Schrödinger equations with separation of variables. More interesting choices $F_1 = x_1^2$ and $F_1 = x_1^{2/3}$ lead, respectively, to potentials (l-arbitrary real constant):

$$V^{(1,2)} = \mp 2\hbar x_2 + lx_1^{-2} + \tfrac{1}{2}\left(x_1^4 + 6x_1^2x_2^2 + 8x_2^4\right) - \tfrac{1}{8}m_2\left(x_1^2 + 4x_2^2\right) \qquad (37)$$

$$V^{(1,2)} = \tfrac{7}{36}\hbar^2x_1^{-2} \pm \tfrac{2}{9}\hbar x_2x_1^{-4/3} + \tfrac{1}{9}x_1^{-2/3}\left(x_2^2 + \tfrac{9}{2}x_1^2\right)9lx_1^{2/3} - \tfrac{1}{8}m_2\left(9x_1^2 + 4x_2^2\right). \qquad (38)$$

(2) $G_1 \neq 0$.

(2a) For $F_1 = k_1 \neq 0$ equation (31) leads to the following equation for $G_1(x_1)$:

$$\int \frac{G_1^2\,\mathrm{d}G_1}{\sqrt{k - \frac{1}{2}m_2G_1^4}} = x_1$$

which has the solutions in terms of elementary functions in two cases: when $k > 0, m_2 = 0$ or $k = 0, m_2 < 0$. For the first one, after redefinition of constants and translation in x_2,

$$V^{(1,2)} = -\tfrac{5}{36}\hbar^2 x_1^{-2} + \tfrac{1}{3}\hbar k_2(1 \pm 1)x_1^{-2/3} - \tfrac{1}{3}k_1k_2x_2x_1^{-2/3} + \frac{1}{4}\left(k_2^2 + \frac{3k_1m_1}{k_2}\right)x_1^{2/3} - m_1x_2 + \frac{k_1^2}{2}. \quad (39)$$

The second case ($k = 0, m_2 < 0$) leads to separation of variables.

(2b) If $F_1 = x_1^{2/3}$ and $F_1 = x_1^2$ the general solutions of equation (32) can be found and after substitution into (30), (31) give the function $K_1(x_1)$. Corresponding potentials are

$$V^{(1,2)} = \tfrac{7}{36}\hbar^2 x_1^{-2} \pm \tfrac{2}{9}\hbar x_2 x_1^{-4/3} \pm \hbar k_1 + \tfrac{1}{9}x_1^{-2/3}\left(x_2^2 + \tfrac{9}{2}x_1^2\right) - \tfrac{5}{3}k_1x_2x_1^{2/3} + \frac{3m_1}{8k_1}x_1^{2/3} + \tfrac{1}{4}k_1^2x_1^2 - m_1x_2 + \tfrac{1}{9}k_1(1 + 15k_1)x_2^2 \quad (40)$$

$$V^{(1,2)} = \left(\tfrac{3}{4}\hbar^2 \mp \hbar k_1 + \tfrac{1}{4}k_1^2\right)x_1^{-2} \mp 2\hbar x_2 + \tfrac{1}{2}\left(x_1^4 + 6x_2^2x_1^2 + 8x_2^4\right) - \tfrac{1}{8}(12k^2 + m_2)(x_1^2 + 4x_2^2) - \tfrac{1}{4}\left(km_2 + 6k^3 + 4k_1\right)x_2. \quad (41)$$

Because all potentials (37)–(41) are singular at the x_2 axis, similar to the case of Lorentz metrics, it is necessary to investigate separately the behaviour of wavefunctions at $x_1 \to 0$. Straightforward though cumbersome calculations show that the fourth-order differential operators $R^{(1)}, R^{(2)}$ (see equations (3)) preserve the asymptotics of wavefunctions for systems (37)–(41) and play the role of true dynamical symmetry operators.

3. Construction of 2D integrable classical systems by the limit $\hbar \to 0$

Quantum dynamical symmetries which were found by the intertwining method in section 2 have their natural analogues (integrals of motion) in the corresponding classical systems. These integrals of motion are polynomials of fourth order in momenta. Similar to section 2, it is useful to consider separately the classical limits for Lorentz and degenerate metrics.

3.1. Lorentz metrics

For the Lorentz metrics in the limit $\hbar \to 0$ classical supercharge functions become

$$q_{cl}^{\pm} = -4p_+p_- \pm \mathrm{i}(C_-(x_-)p_+ + C_+(x_+)p_-) + B(x_-, x_+).$$

From equations (8) and (3) we find that the classical Hamiltonian

$$h_{cl} = 2(p_+^2 + p_-^2) + \tfrac{1}{8}(C_+^2 + C_-^2) + \tfrac{1}{4}[F_2(x_+ - x_-) - F_1(x_+ + x_-)] \quad (42)$$

has the additional integral of motion:

$$I = 16p_+^2p_-^2 + C_+^2p_-^2 + C_-^2p_+^2 - 2(F_1 + F_2)p_+p_- + B^2. \quad (43)$$

Such types of classical systems were considered in the literature (see [13] and references therein). Usually the functional equation, which provides existence of integrals of motion for corresponding classical systems, is solved by the Lax method. Comparison of (42) and (43) with notations in [13] leads to the relations

$$\begin{aligned} v_1 &\equiv 1/8C_+^2(x_1) \\ v_2 &\equiv 1/8C_-^2(x_2) \\ v_3(x_-) &\equiv 1/16F_2(x_-) \\ v_4(x_+) &\equiv -1/16F_1(x_+) \end{aligned} \quad (44)$$

and functions v_k must satisfy [13] a certain functional equation (see equation (47) below). In the next section we prove that (44), where $C_\pm$, $F_{1,2}$ are solutions of the system (6) and (7), also satisfy equation (47). Moreover, some additional solutions of this equation will be found in section 4.

3.2. Degenerate metrics

Let us study the integrable classical systems which can be obtained in the limit $\hbar \to 0$ from SSQM systems with degenerate metrics in $q^\pm$. The classical supercharges have the form $q^+_{cl} = (q^-_{cl})^* = -p_1^2 + \mathrm{i}C_k(\vec{x})p_k + B(\vec{x})$ and the Hamiltonian

$$H_{cl} = p_k^2 + \tfrac{1}{2}x_2^2\partial_1^2 F_1^2 - x_2(F_1G_1)' + K_1(x_1) + K_2(x_2) \tag{45}$$

has the integral of motion of fourth order in momenta:

$$I \equiv q^+_{cl}q^-_{cl} = p_1^4 + (C_1^2 - 2B)p_1^2 + C_2^2p_2^2 + 2C_1C_2p_1p_2 + B^2. \tag{46}$$

All functions in (45) and (46) were defined in the previous section, where we have to put $\hbar = 0$.

Thus in the case of degenerate metrics the following classical integrable systems are obtained (new definitions of constants were used for some of these systems):

(1)

$$V = \tfrac{1}{2}\left(x_1^4 + 6x_1^2x_2^2 + 8x_2^4\right) + m\left(x_1^2 + 4x_2^2\right) + lx_1^{-2}$$
$$I = p_1^4 + \left(6x_1^2x_2^2 + mx_1^2 + x_1^4 + 2lx_1^{-2}\right)p_1^2 + x_1^4p_2^2 - 4x_1^3x_2p_1p_2 + \left(x_1^2x_2^2 + mx_1^2 + \tfrac{1}{2}x_1^4 + lx_1^{-2}\right)^2$$

(2)

$$V = \tfrac{1}{9}x_1^{-2/3}\left(\tfrac{9}{2}x_1^2 + x_2^2\right) + m\left(9x_1^2 + 4x_2^2\right) + kx_1^{2/3}$$
$$I = p_1^4 + \left(\tfrac{2}{9}x_2^2x_1^{-2/3} + 2kx_1^{2/3} + 18mx_1^2 + x_1^{4/3}\right)p_1^2 + x_1^{4/3}p_2^2 - \tfrac{4}{3}x_2x_1^{1/3}p_1p_2 + \left(\tfrac{1}{9}x_2^2x_1^{-2/3} - kx_1^{2/3} - 9mx_1^2 - \tfrac{1}{2}x_1^{4/3}\right)^2$$

(3)

$$V = -\frac{k_1k_2}{3}x_2x_1^{-2/3} + \frac{1}{4}\left(k_2^2 + \frac{3k_1m_1}{k_2}\right)x_1^{2/3} - m_1x_2$$
$$I = p_1^4 + \left[\frac{1}{2}\left(k_2^2 + \frac{3k_1m_1}{k_2}\right)x_1^{2/3} - \frac{2k_1k_2}{3}x_2x_1^{-2/3} + k_1^2\right]p_1^2 + k_1^2p_2^2 + 2k_1k_2x_1^{1/3}p_1p_2 + \left[\frac{k_1k_2}{3}x_2x_1^{-2/3} + \frac{1}{4}\left(k_2^2 - \frac{3k_1m_1}{k_2}\right)x_1^{2/3} - \frac{k_1^2}{2}\right]^2$$

(4)

$$V = \frac{1}{9}x_1^{-2/3}\left(\frac{9}{2}x_1^2 + x_2^2\right) - \frac{5k}{3}x_2x_1^{2/3} + \frac{3m}{8k}x_1^{2/3} + \frac{k^2}{4}x_1^2 - mx_2 + \frac{k}{9}(1 + 15k)x_2^2$$
$$I = p_1^4 + \left(\frac{2}{9}x_2^2x_1^{-2/3} - \frac{10k}{3}x_2x_1^{2/3} + \frac{k^2}{2}x_1^2 + \frac{3m_1}{4k}x_1^{2/3} + x_1^{4/3}\right)p_1^2 + x_1^{4/3}p_2^2 + 2x_1^{2/3}\left(-\tfrac{2}{3}x_2x_1^{-1/3} + kx_1\right)p_1p_2 + \left(\frac{1}{9}x_2^2x_1^{-2/3} + kx_2x_1^{2/3} - \frac{3m_1}{8k}x_1^{2/3} - \frac{1}{2}x_1^{4/3} + \frac{k^2}{4}x_1^2\right)^2$$

(5)

$$
\begin{aligned}
V &= \tfrac{1}{2}\big(x_1^4 + 6x_1^2x_2^2 + 8x_2^4\big) + m\big(x_1^2 + 4x_2^2\big) + \tfrac{1}{4}k_1^2x_1^{-2} - \big(k_1 - 2mk - \tfrac{3}{2}k^3\big)x_2 \\
I &= p_1^4 + \big(6x_1^2x_2^2 + 2\big(m + k^2\big)x_1^2 - 2k_1x_2 - 4kx_1^2x_2 + x_1^4 + \tfrac{1}{2}k_1^2x_1^{-2} + kk_1\big)p_1^2 \\
&\qquad + x_1^4p_2^2 + 2x_1\big(k_1 - 2x_1^2x_2\big)p_1p_2 \\
&\qquad + \big(x_1^2x_2^2 + \big(m + \tfrac{1}{2}k^2\big)x_1^2 + \tfrac{1}{2}x_1^4 + k_1x_2 - \tfrac{}{4}k_1^2x_1^{-2} - \tfrac{1}{2}kk_1\big)^2.
\end{aligned}
$$

These potentials are not new: they were found by other methods in [14–16].

In conclusion of this section we formulate the procedure of construction of integrals of motion in terms of classical mechanics objects. For any classical Hamiltonians H_{cl} and a complex function $q_{cl}^+(\vec{x},\vec{p}) = (q_{cl}^-)^*$, polynomial in momenta, such that

$$
\{q_{cl}^+, H_{cl}\} = \mathrm{i}f(\vec{x},\vec{p})q_{cl}^+ \qquad \{(q_{cl}^+)^*, H_{cl}\} = -\mathrm{i}f(\vec{x},\vec{p})(q_{cl}^+)^*
$$

with arbitrary real function $f(\vec{x},\vec{p})$, the classical factorizable integral of motion $I = q_{cl}^+ \cdot q_{cl}^-$ exists ($\{\,,\,\}$-Poisson brackets).

4. Integrable systems connected with the Lax method

Let us consider classical systems with potentials of the form

$$
V(x_1, x_2) = v_1(x_1) + v_2(x_2) + v_3(x_1 - x_2) + v_4(x_1 + x_2).
$$

It is known [13] that these systems have the integrals of motion of fourth order in momenta:

$$
I = \tfrac{1}{2}p_1^2p_2^2 + v_2(x_2)p_1^2 - \big(v_3(x_1 - x_2) - v_4(x_1 + x_2)\big)p_1p_2 + v_1(x_1)p_2^2 + f
$$

if the functions v_1, v_2, v_3, v_4 satisfy the basic functional equation:

$$
\begin{aligned}
&[v_4(x_1 + x_2) - v_3(x_1 - x_2)][v_2''(x_2) - v_1''(x_1)] + 2[v_4''(x_1 + x_2) - v_3''(x_1 - x_2)] \\
&\qquad \times [v_2(x_2) - v_1(x_1)] + 3v_4'(x_1 + x_2)[v_2'(x_2) - v_1'(x_1)] \\
&\qquad + 3v_3'(x_1 - x_2)[v_2'(x_2) + v_1'(x_1)] = 0.
\end{aligned} \tag{47}
$$

There is a list of known particular solutions of equation (47) in the book [13]. To search for new solutions of this equation it is useful to rewrite it in the equivalent form

$$
\partial_2(vv_2' + 2v_2\partial_2 v) = \partial_1(vv_1' + 2v_1\partial_1 v) \tag{48}
$$

where

$$
v \equiv v_4(x_1 + x_2) - v_3(x_1 - x_2). \tag{49}
$$

The general solution of equation (48) is

$$
v = \left[G\left(\int \frac{\mathrm{d}x_1}{\sqrt{v_1}} + \int \frac{\mathrm{d}x_2}{\sqrt{v_2}}\right) + L\left(\int \frac{\mathrm{d}x_1}{\sqrt{v_1}} - \int \frac{\mathrm{d}x_2}{\sqrt{v_2}}\right)\right]\Big/\sqrt{v_1v_2} \qquad v_1 \not\equiv 0 \qquad v_2 \not\equiv 0 \tag{50}
$$

where G and L are arbitrary functions of their arguments. Thus the problem is reduced to searching for the functions v of the form (50), which satisfy the condition

$$
(\partial_1^2 - \partial_2^2)v = 0. \tag{51}
$$

In particular, it is easy to check that all solutions which were found from SSQM in the $\hbar \to 0$ limit in section 3 (see equation (44)) are particular solutions of equation (47) and have the form (50) with $G \equiv 0$.

Let us apply the technique, which was used in investigation of the system (6) and (7) in the framework of SSQM, to find the new particular solutions of equation (48).

(1) If the function v is factorizable $v = u_1(x_1) \cdot u_2(x_2)$, equation (48) admits separation of variables and its solutions have the form

$$\begin{aligned} v_k &= \frac{n_k[a_k \exp(\sqrt{\lambda} x_k) + b_k \exp(-\sqrt{\lambda} x_k)] + l_k}{(a_k \exp(\sqrt{\lambda} x_k) - b_k \exp(-\sqrt{\lambda} x_k))^2} \qquad k = 1, 2. \\ v_3 &= a_1 b_2 \exp(\sqrt{\lambda} \cdot x_-) + a_2 b_1 \exp(-\sqrt{\lambda} \cdot x_-) \\ v_4 &= a_1 a_2 \exp(\sqrt{\lambda} \cdot x_+) + b_1 b_2 \exp(-\sqrt{\lambda} \cdot x_+) \end{aligned} \tag{52}$$

where for $\lambda > 0$ all constants are real and for $\lambda < 0$ $a_k = b_k^*$ and n_k, l_k are real.

(2) Let us introduce new functions

$$v_1 \equiv (W_1')^{-2} \qquad v_2 \equiv (W_2')^{-2}. \tag{53}$$

Then equation (51) means that (derivatives of G and L are taken in their arguments):

$$\begin{aligned} (G + L)\left[\frac{W_1'''}{W_1'} - \frac{W_2'''}{W_2'}\right] &+ 3[(W_1'' - W_2'')G' + (W_1'' + W_2'')L'] \\ &+ (W_1'^2 - W_2'^2)(G'' + L'') = 0. \end{aligned} \tag{54}$$

It is possible to construct several particular solutions of (54).

(2a)
$$W_{1,2} = \sigma_{1,2} \exp(\lambda x_{1,2}) + \delta_{1,2} \exp(-\lambda x_{1,2}) \tag{55}$$

where constant λ^2 is real and σ_k, δ_k are complex. If these constants satisfy the condition

$$\sigma_1 \delta_1 = \sigma_2 \delta_2 \tag{56}$$

then substitution of (55) into (54) leads to an equation with separable variables. Its solutions are

$$G(W_1 + W_2) = \frac{\alpha_1}{(W_1 + W_2)^2} + \alpha(W_1 + W_2)^2 + \beta_1 \tag{57}$$

$$L(W_1 - W_2) = \frac{\alpha_2}{(W_1 - W_2)^2} + \alpha(W_1 - W_2)^2 + \beta_2 \tag{58}$$

with arbitrary constants $\alpha, \alpha_i, \beta_i (i = 1, 2)$.

Let us consider the case with $\lambda^2 > 0$. Constants σ_1 and δ_1 must be both real or both positive because of the requirement that functions $v_n(x_n)$ for $n = 1, 2$ should be real. Analogous arguments work for the pair σ_2, δ_2. The condition (56) for real (σ_i, δ_i) leads to two options:

$$W_1(x) = W_2(x) = k \cosh(\lambda x) \tag{59}$$

$$W_1(x) = W_2(x) = k \sinh(\lambda x) \qquad k \in \mathbb{R} \tag{60}$$

and the function v is real for real constants $\alpha, \alpha_i, \beta_i (i = 1, 2)$.

In the case of real (σ_1, δ_1) and imaginary (σ_2, δ_2) the solutions take the form

$$W_1 = k \sinh(\lambda x_1) \qquad W_2 = \mathrm{i}k \cosh(\lambda x_2) \tag{61}$$

$$W_1 = k \cosh(\lambda x_1) \qquad W_2 = \mathrm{i}k \sinh(\lambda x_2) \qquad k \in \mathbb{R}. \tag{62}$$

Let us remark that in this case the arguments of the functions G and L are complex conjugated and in order to have a real function v the following relation must be fulfilled:

$$L^*(W_1 + W_2) = -G(W_1 + W_2).$$

Therefore, $\alpha_2 = -\alpha_1^*$; $\beta_2 = -\beta_1^*$; $\alpha = \alpha^*$ in (57) and (58).

(2b) $$W_1 = W_2 = gx^2 \qquad g^2 \in \mathbb{R}. \tag{63}$$

It follows from equation (54) that

$$G = a_0 + a_1(W_1 + W_2) + a(W_1 + W_2)^2 \qquad L = b_0 - \tfrac{1}{4}a(W_1 - W_2)^2 + \frac{b}{(W_1 + W_2)^2}$$

where all constants are real.

Thus the functions v_n $(n = 1, \ldots, 4)$ for solutions (59)–(63) are, respectively,

$$\begin{aligned} v_1(x) &= v_2(x) = \frac{k}{\sinh^2(\lambda x)} \\ v_3(x) &= v_4(x) = \frac{k_1}{\sinh^2(\lambda x)} + \frac{k_2}{\sinh^2(\lambda x/2)} + k_3\cosh(2\lambda x) + k_4\cosh(\lambda x) \end{aligned} \tag{64}$$

$$\begin{aligned} v_1(x) &= v_2(x) = \frac{k}{\cosh^2(\lambda x)} \\ v_3(x) &= \frac{k_1}{\sinh^2(\lambda x)} + \frac{k_2}{\sinh^2(\lambda x/2)} + k_3\cosh(2\lambda x) + k_4\cosh(\lambda x) \\ v_4(x) &= \frac{k_1 + 4k_2}{\sinh^2(\lambda x)} - \frac{k_2}{\sinh^2(\lambda x/2)} + k_3\cosh(2\lambda x) - k_4\cosh(\lambda x) \end{aligned} \tag{65}$$

$$\begin{aligned} v_1(x) &= \frac{k}{\cosh^2(\lambda x)} \qquad v_2(x) = -\frac{k}{\sinh^2(\lambda x)} \\ v_3(x) &= v_4(x) = \frac{k_1 + k_2\sinh(\lambda x)}{\cosh^2(\lambda x)} + k_3\cosh(2\lambda x) - k_4\sinh(\lambda x) \end{aligned} \tag{66}$$

$$\begin{aligned} v_1(x) &= \frac{k}{\sinh^2(\lambda x)} \qquad v_2(x) = -\frac{k}{\cosh^2(\lambda x)} \\ v_3(x) &= \frac{k_1 + k_2\sinh(\lambda x)}{\cosh^2(\lambda x)} + k_3\cosh(2\lambda x) + k_4\sinh(\lambda x) \\ v_4(x) &= \frac{k_1 - k_2\sinh(\lambda x)}{\cosh^2(\lambda x)} + k_3\cosh(2\lambda x) - k_4\sinh(\lambda x) \end{aligned} \tag{67}$$

$$v_1(x) = v_2(x) = kx^{-2} \qquad v_3(x) = v_4(x) = k_1x^{-2} + k_2x^2 + k_3x^4 + k_4x^6 \tag{68}$$

$$\begin{aligned} v_1 &= \frac{g_1}{(\delta_1\exp(\lambda x) - \sigma_1\exp(-\lambda x))^2} \qquad v_2 = \frac{g_2}{(\delta_2\exp(\lambda x) - \sigma_2\exp(-\lambda x))^2} \\ v_3(x) &= k_1(\delta_1\sigma_2\exp(\lambda x) + \sigma_1\delta_2\exp(-\lambda x)) + k_2(\delta_1^2\sigma_2^2\exp(2\lambda x) + \sigma_1^2\delta_2^2\exp(-2\lambda x)) \\ v_4(x) &= k_1(\delta_1\delta_2\exp(\lambda x) + \sigma_1\sigma_2\exp(-\lambda x)) + k_2(\sigma_1^2\sigma_2^2\exp(2\lambda x) + \delta_1^2\delta_2^2\exp(-2\lambda x)). \end{aligned} \tag{69}$$

The solutions (65)–(67) are absent in the list of [13] and to our knowledge they are novel. As to expressions (69), they are present in [13] only for $\sigma_1\delta_1 > 0,\ \sigma_2\delta_2 > 0$.

In conclusion, let us note that one can easily check some invariance properties of equation (47). In particular, from arbitrary solutions (52) and (64)–(69) one derives new ones with

$$v_4(2x) \to v_1(x) \qquad v_3(2x) \to v_2(x) \qquad v_1(x) \to -v_3(x) \qquad v_2(x) \to -v_4(x). \tag{70}$$

Equation (47) is invariant if $v_{1,2} \to v_{1,2} + c$ and $v_{3,4} \to v_{3,4} + \tilde{c}$ with arbitrary constants $c, \tilde{c}$. It is also invariant under dilation of all arguments $x_i \to \Lambda x_i$.

5. Integrable systems with potentials, expressed in elliptic functions

In this section we formulate the method of construction of new integrable systems from solutions v_i of equation (47), which were found in the previous section. If we define new functions

$$W_1'^2 = f_1(W_1) \qquad W_2'^2 = f_2(W_2) \tag{71}$$

equation (54) takes the form

$$\begin{aligned}[G(W_1+W_2) - L(W_1-W_2)][f_2''(W_2) - f_1''(W_1)] \\ +2[G''(W_1+W_2) - L''(W_1-W_2)][f_2(W_2) - f_1(W_1)] \\ +3G'(W_1+W_2)[f_2'(W_2) - f_1'(W_1)] \\ +3L'(W_1-W_2)[f_2'(W_2) + f_1'(W_1)] = 0.\end{aligned} \tag{72}$$

This equation has the same structure as equation (47), thus from (71) we obtain

$$\begin{aligned}&W_1'^2 = v_1(W_1) \qquad W_2'^2 = v_2(W_2) \\ &G(W_1+W_2) = v_4(W_1+W_2) \qquad L(W_1-W_2) = -v_3(W_1-W_2)\end{aligned} \tag{73}$$

where we can use solutions (52) and (64)–(69) for $v_n(x)$ $(n = 1, \ldots, 4)$ with arguments $x_{1,2}$ replaced, respectively, by $W_{1,2}$. The first pair of equations in (73) defines functions $W_1(x_1)$, $W_2(x_2)$ and the last two equations define new functions G, L. Substituting these sets of functions into (53) and (50), we find some new solutions V_n of the same equation (47) from already known solutions v_n (see (52) and (64)–(69)).

Let us consider several examples of this method of reproducing new solutions.

(1) The first attempt to start our procedure from the simplest solutions (68) leads to a discouraging result: we obtain the same solution. However, we can firstly transform $v_{1,2} \Longleftrightarrow v_{3,4}$ in (68), using the invariance property (70) mentioned at the very end of section 4. As follows from (73), the functions $W_{1,2}(x_{1,2})$ are defined from the equations (we omit indices $i = 1, 2$)

$$W'^2(x) = k_1 W^{-2} + k_2 W^2 + k_3 W^4 + k_4 W^6 + k_0$$

with constant k_i. It is useful to rewrite them in terms of functions $U(x) \equiv \frac{1}{2} W^2(x)$:

$$U'^2(x) = k_1 + k_0 U + k_2 U^2 + k_3 U^3 + k_4 U^4. \tag{74}$$

From (50) and (53) we obtain new solutions:

$$V_1 = g_1 \frac{U(x_1)}{U'^2(x_1)} \qquad V_2 = g_1 \frac{U(x_2)}{U'^2(x_2)} \tag{75}$$

$$V_4(x_1+x_2) - V_3(x_1-x_2) = g \frac{U'(x_1)U'(x_2)}{(U(x_1) - U(x_2))^2} \tag{76}$$

where arbitrary constants g_1, g appear because solutions v_n are defined from equation (47) up to constant factors. We can check directly that for $U(x)$, which satisfy equation (74), the right-hand side of (76) is the solution of equation (51).

When the right-hand side polynomial in equation (74) has degenerate roots, $U(x)$ can be expressed through elementary functions and corresponds to solutions (52) and (64)–(69), found above. When all roots are simple, $U(x)$ can be given in terms of elliptic functions.

(1a) $\qquad U(x) = \wp(x) + b$

where b is constant and $\wp(x)$ is the Weierstrass function with semiperiods ω_1 and ω_2 (their values depend on constants k_i). Equations (75) and (76) lead to new solutions:

$$V_1(x) = V_2(x) = a_1\wp(x+\omega_1) + a_2\wp(x+\omega_2) + a_3\wp(x+(\omega_1+\omega_2)/2)$$
$$V_3(x) = V_4(x) = a\wp(x)$$

where a is an arbitrary constant and constants a_k $(k = 1, 2, 3)$ satisfy the following condition:

$$\sum_k a_k^2 - \sum_{i\neq j} a_i a_j = 1.$$

(1b) $U(x) = \mathrm{sn}\,x + b$, where b is constant and $\mathrm{sn}\,(x)$ is the Jacobi function with modulus k (it depends again on constants k_i). In this case new solutions are

$$V_1(x) = V_2(x) = a\frac{\mathrm{sn}\,x + b}{\mathrm{cn}^2 x\,\mathrm{dn}^2 x}$$
$$V_3(x) = c\left(\frac{1}{\mathrm{sn}^2(x/2)} - k^2\mathrm{cn}^2(x/2) - k^2\right)$$
$$V_4(x) = c(1-k^2)\left(\frac{1}{\mathrm{cn}^2(x/2)} - \frac{1}{\mathrm{sn}^2(x/2)}\right).$$

(1c) $U(x) = \mathrm{dn}x + b$.

Correspondingly, the new solution takes the form

$$V_1(x) = V_2(x) = a\frac{\mathrm{dn}x + b}{\mathrm{sn}^2 x\,\mathrm{dn}^2 x}$$
$$V_3(x) = V_4(x) = c\left(\frac{1}{\mathrm{sn}^2(x/2)} + (1-k^2)\mathrm{cn}^2(x/2)\right).$$

(2) The second solution of (47), which we can take as the starting point of the proposed procedure, is one of solutions in equation (52):

$$v_1(x) = v_2(x) = a\cos x + c \qquad a > 0 \quad c > 0$$
$$v_3(x) = k_1(\sin(x/2))^{-2} + k_2(\sin(x/4))^{-2}$$
$$v_4(x) = k_3(\sin(x/2))^{-2} + k_4(\sin(x/4))^{-2}.$$

Then according to equation (73), functions $W(x)$ must be found from the equation

$$W'^2(x) = a\cos W(x) + c$$

the solution of which can be expressed through the Jacobi function with modulus k $(k^2 \equiv 2a/(a+c))$:

$$W(x) = \arccos(1 - 2(k\,\mathrm{sn}\,y)^{-2}) \qquad y \equiv \tfrac{1}{2}\sqrt{(a+c)}\,x. \tag{77}$$

Thus the new solution of equation (47) is

$$V_4(x_1+x_2) = a_1\mathrm{sn}^2(y_+/2) + a_2\frac{\mathrm{cn}^2(y_+/2)}{\mathrm{dn}^2(y_+/2)} + a_3\frac{\mathrm{dn}^2(y_+/2)}{\mathrm{sn}^2(y_+/2)\mathrm{cn}^2(y_+/2)} + a_4\frac{\mathrm{dn}^2(y_+/2)}{\mathrm{cn}^2(y_+/2)}$$
$$V_3(x_1-x_2) = a_2\mathrm{sn}^2(y_-/2) + a_1\frac{\mathrm{cn}^2(y_-/2)}{\mathrm{dn}^2(y_-/2)} + a_3\frac{\mathrm{dn}^2(y_-/2)}{\mathrm{sn}^2(y_-/2)\mathrm{cn}^2(y_-/2)} + a_4\frac{1}{\mathrm{sn}^2(y_-/2)}$$
$$V_1(x) = V_2(x) = \frac{a_0}{\mathrm{cn}^2 y}$$

where a_0 and a_k are arbitrary constants.

In conclusion, we briefly mention the analogous method of construction of new integrable systems in quantum mechanics (see section 2). In this case the procedure is based on equation (11) of section 2. Similarly to (71), we introduce in (11) new functions $M_\pm(A_\pm)$:

$$A_+'^2(x_+) = M_+(A_+) \qquad A_-'^2(x_-) = M_-(A_-). \tag{78}$$

Then equation (11) takes the form

$$[M_+''(A_+) - M_-''(A_-)]L(A_+ - A_+) + 3[M_+'(A_+) + M_-'(A_-)]L'(A_+ - A_+) \\ +2[M_+(A_+) - M_-(A_-)]L''(A_+ - A_+) = 0.$$

The general solution of this equation can be found in [17] and the corresponding functions $M_\pm(A_\pm)$, $L(A_+ - A_-)$ can be used here to find $A_\pm(x)$ from (78). Substitution of these functions $A_\pm(x)$ into $M_\pm(A_\pm)$, $L(A_+ - A_-)$ and equation (10) leads to new solutions of the system equations (6) and (7) for the quantum case.

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